On the Stability of Low Pass Graph Filter with a Large Number of Edge Rewires

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Motivation

Graph convolutional neural networks (GCNs) performs very well empirically. Graph filters can be viewed as a building block of GCNs, hence stability properties of graph filters can characterize stability of GCNs. However, theoretical stability of graph filter/GCNs is not fully understood. Existing works *mostly focus on stability w.r.t. perturbation size*, i.e. small perturbation regime.

Objective

We investigate the conditions for stability graph filter when the graph is **subject to a large number of perturbations** (in this work, edge rewiring). Specifically, we **connect the stability bound to community structure**, allowing number of perturbations to be arbitrarily large.

Preliminaries

Undirected, connected graph. $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has n nodes, with adjacency $\mathbf{A}$ and degree $\mathbf{D}$

Two GSOs. $\mathbf{S} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T$

- Unnormalized Laplacian $\mathbf{L}_U = \mathbf{D} - \mathbf{A}$
- Normalized Laplacian $\mathbf{L}_{\mathsf{norm}} = \mathbf{D}^{-1/2}\mathbf{L}_U\mathbf{D}^{-1/2}$

Eigenvalues of $\mathbf{S}$: $0 = \lambda_1 \leq \ldots \leq \lambda_n$

Perturbation. Consider a perturbation of $\mathcal{G}$ (with GSO $\mathbf{S}$) into $\hat{\mathcal{G}}$ (with GSO $\hat{\mathbf{S}}$)

Graph Filter.

$$\mathcal{H}(\mathbf{S}) = \sum_{t=0}^{T-1} h_t \mathbf{S}^t = \mathbf{V} h(\boldsymbol{\Lambda}) \mathbf{V}^T$$

Def. 1. Low pass graph filter

$\mathcal{H})(\cdot)$ is $(\underline{\lambda}, \overline{\lambda})$-low pass if

$$\eta = \Big(\min_{\lambda \in [0, \underline{\lambda}]} |h(\lambda)| \Big)^{-1} \max_{\lambda \in [\overline{\lambda}, \infty)} |h(\lambda)| < 1$$

where $\underline{\lambda} \leq \overline{\lambda}$; η is called low pass ratio. Cutoff frequency $\underline{\lambda}$ may depend on graph size n.

Measure of stability $\mathbb{D}_{\mathcal{H}}(\mathbf{S}, \hat{\mathbf{S}})$

The graph filter distance between $\mathcal{H}(\mathbf{S})$ and $\mathcal{H}(\hat{\mathbf{S}})$ is defined similar to [1] and [2]:

$$\mathbb{D}_{\mathcal{H}}(\mathbf{S}, \hat{\mathbf{S}}) = \sup_{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}} \frac{\left\| \mathcal{H}(\mathbf{S})\mathbf{x} - \mathcal{H}(\hat{\mathbf{S}})\mathbf{x} \right\|_2}{\|\mathbf{x}\|_2} = \|\mathcal{H}(\mathbf{S}) - \mathcal{H}(\hat{\mathbf{S}})\|_2$$

We ignore node permutation for simplicity.

Bounding $\mathbb{D}_{\mathcal{H}}(\mathbf{S}, \hat{\mathbf{S}})$

Assumptions.

1. $\sup_{\lambda \in [0, \underline{\lambda}]} |h(\lambda)| \leq \mathbb{H}_{\max}$
2. $|h(\lambda) - h(\lambda')| \leq L_{\mathbb{H}} |\lambda - \lambda'|, \ \forall\, \lambda, \lambda' \in [0, \underline{\lambda}]$

Notations. $\boldsymbol{\Lambda}_k$ and $\mathbf{V}_k$ (resp. $\hat{\boldsymbol{\Lambda}}_k$ and $\hat{\mathbf{V}}_k$) as matrices of smallest-k eigenvalues and eigenvectors of $\mathbf{S}$ (resp. $\hat{\mathbf{S}}$)

Spectral gap requirement. Graph frequencies of $\mathbf{S}, \hat{\mathbf{S}}$ satisfy $\lambda_k \leq \underline{\lambda} < \overline{\lambda} \leq \lambda_{k+1}$ and $\hat{\lambda}_k \leq \underline{\lambda} < \overline{\lambda} \leq \hat{\lambda}_{k+1}$ for some $1 \leq k \leq n-1$. The common transition region $(\overline{\lambda}, \underline{\lambda})$ separates low and high frequencies of the two graphs.

Thm. 1. Upperbound of $\mathbb{D}_{\mathcal{H}}(\mathbf{S}, \hat{\mathbf{S}})$

If $\mathcal{H}(\cdot)$ be $(\overline{\lambda}, \underline{\lambda})$ low pass filter with ratio η, then

$$\mathbb{D}_{\mathcal{H}}(\mathbf{S}, \hat{\mathbf{S}}) \leq 2\mathbb{H}_{\max}\eta + L_{\mathbb{H}} \|\boldsymbol{\Lambda}_k - \hat{\boldsymbol{\Lambda}}_k\|_2 + 2\mathbb{H}_{\max} \|\mathbf{V}_k - \hat{\mathbf{V}}_k\|_2$$

Interpreting Thm. 1. The bound of $\mathbb{D}_{\mathcal{H}}(\mathbf{S}, \hat{\mathbf{S}})$ consists of three terms

- First term depends on frequency response
- Last two terms capture similarity in community structure (like spectral clustering)

Thm. 1 implies that low pass graph filters $\mathcal{H}(\mathbf{S})$, $\mathcal{H}(\hat{\mathbf{S}})$ (with $\eta \ll 1$) has $\mathbb{D}_{\mathcal{H}}(\mathbf{S}, \hat{\mathbf{S}})$ *dependent on dissimilarity between their community structure*, hence insensitive to number of edge rewires.

Stability with Edge Rewiring

We apply the result to *planted partition model (PPM) with structural invariant edge rewiring.*

$\mathsf{PPM}(n, k, a_n, b_n)$ has n nodes partitioned into k equal blocks; the matrix of connection probability of two nodes from two blocks is $\mathbf{B} = [b_{ij}]_{1 \leq i,j \leq k} = a_n \mathbf{I} + b_n \mathbf{1}\mathbf{1}^T$.

Edge rewiring scheme.

1. Generate original $\mathcal{G} \sim \mathsf{PPM}(n, k, a_n, b_n)$
2. For each block (i, j), randomly (i) delete a portion of $p_{\mathsf{re}} \in [0, 1]$ edges, and (ii) add edges to unconnected node pairs with probability $[b_{ij}^{-1} - (1 - p_{\mathsf{re}})]^{-1} p_{\mathsf{re}}$ independently; we obtain perturbed graph $\hat{\mathcal{G}} \sim \mathsf{PPM}(n, k, a_n, b_n)$

Stability with $\mathbf{L}_U$

Let $\mathcal{G}, \hat{\mathcal{G}} \sim \mathsf{PPM}(n, 2, \alpha \log n/n, \beta \log n/n)$ with $\mathbf{L}_U$, $\hat{\mathbf{L}}_U$. Suppose that $\sqrt{\alpha} - \sqrt{\beta} > \sqrt{2}$, then with high probability,

$$\|\mathbf{V}_2 - \hat{\mathbf{V}}_2\|_2 = o(1), \ \|\boldsymbol{\Lambda}_2 - \hat{\boldsymbol{\Lambda}}_2\|_2 = \mathcal{O}(\log n/n).$$

Consequently, with high probability,

$$\mathbb{D}_{\mathcal{H}}(\mathbf{L}_U, \hat{\mathbf{L}}_U) \leq 2\eta\, \mathbb{H}_{\max} + \mathbb{H}_{\max}\, o(1) + L_{\mathbb{H}}\, \mathcal{O}(\log n/n).$$

Stability with $\mathbf{L}_{\mathsf{norm}}$

Let $\mathcal{G}, \hat{\mathcal{G}} \sim \mathsf{PPM}(n, k, \alpha \log n/n, \beta \log n/n)$ with $\mathbf{L}_{\mathsf{norm}}$, $\hat{\mathbf{L}}_{\mathsf{norm}}$. Then, with high probability,

$$\|\mathbf{L}_{\mathsf{norm}} - \hat{\mathbf{L}}_{\mathsf{norm}}\|_2 = \mathcal{O}\left(1/\sqrt{\log n}\right)$$

Consider $k = 2$ with $\sqrt{\alpha} - \sqrt{\beta} > \sqrt{2}$ and (for simplicity) $\beta \leq 1$. If $\mathcal{H}(\cdot)$ is $(\underline{\lambda}, \overline{\lambda})$-low pass with $\underline{\lambda} \approx 1/2$, $\overline{\lambda} \approx 1$, then with high probability,

$$\mathbb{D}_{\mathcal{H}}(\mathbf{L}_{\mathsf{norm}}, \hat{\mathbf{L}}_{\mathsf{norm}}) \leq 2\eta\, \mathbb{H}_{\max} + (\mathbb{H}_{\max} + L_{\mathbb{H}})\, \mathcal{O}\left(1/\sqrt{\log n}\right)$$

Prior works: Using Proposition 1 from [2], we can derive a stability bound for $\mathbf{L}_{\mathsf{norm}}$

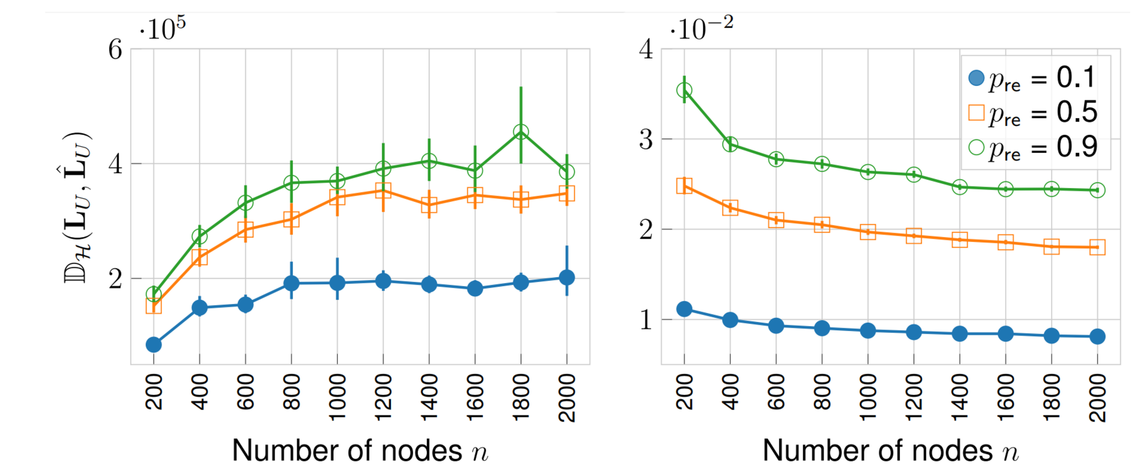
$$\mathbb{D}_{\mathcal{H}}(\mathbf{L}_{\mathsf{norm}}, \hat{\mathbf{L}}_{\mathsf{norm}}) \leq \sum_{t=0}^{T-1} t\, 2^{t-1} |h_t|\, \|\mathbf{L}_{\mathsf{norm}} - \hat{\mathbf{L}}_{\mathsf{norm}}\|_2$$

Numerical Experiments

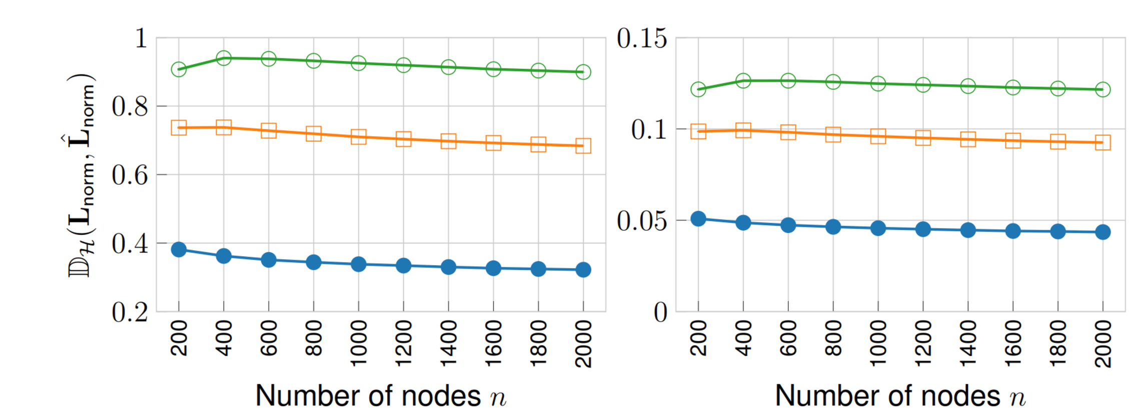
Synthetic experiment.

	$\mathbf{L}_U$	$\mathbf{L}_{\mathsf{norm}}$
$\mathcal{H}_{\mathsf{LP}}(\cdot)$	$\exp(-(1/\log n)\mathbf{L}_U)$	$\exp(-\mathbf{L}_{\mathsf{norm}})$
$\mathcal{H}_{\mathsf{HP}}(\cdot)$	$\exp((1/\log n)\mathbf{L}_U)$	$\exp(\mathbf{L}_{\mathsf{norm}})$

We compare $\mathbb{D}_{\mathcal{H}}(\mathbf{S}, \hat{\mathbf{S}})$ against no. of nodes under different filters. We implement the defined edge rewiring scheme. The figures compare average $\mathbb{D}_{\mathcal{H}}(\mathbf{S}, \hat{\mathbf{S}})$ of (Left) High pass filter $\mathcal{H}_{\mathsf{HP}}(\cdot)$ (Right) Low pass filter $\mathcal{H}_{\mathsf{LP}}(\cdot)$:

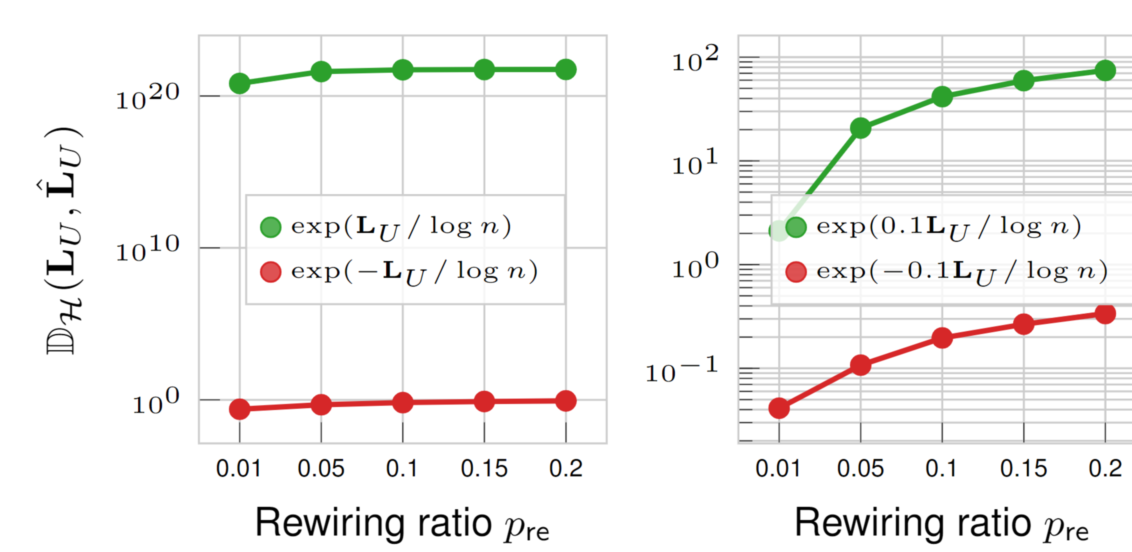


(a) Unnormalized Laplacian $\mathbf{L}_U$ as GSO.



(b) Normalized Laplacian $\mathbf{L}_{\mathsf{norm}}$ as GSO.

Real Data Experiment. We perform edge rewiring on `email-Eu-core` network. The figures compare $\mathbb{D}_{\mathcal{H}}(\mathbf{S}, \hat{\mathbf{S}})$ against rewiring ratios with (Red) Low pass filter (Green) High pass filter:



References

[1] Gama et al., "Stability Properties of Graph Neural Networks", IEEE TSP, 2020

[2] Kenlay et al., "Interpretable Stability Bounds for Spectral Graph Filters", ICML, 2021